

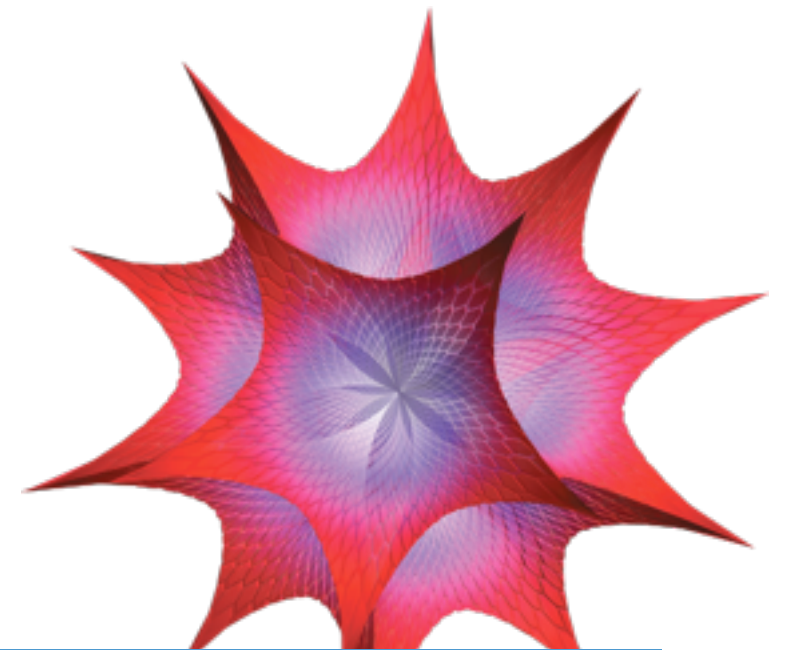
# Algorithmic Variations on Linear Differential Equations

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Inria & ENS de Lyon



# Computer Algebra

- Systems with millions of users
- A scientific area: effective mathematics and their complexity
- 30 years of progress in mathematical algorithms



Thesis in this talk: linear differential equations are a good data-structure.

# Menu dégustation

1. Equations as a data-structure
2. Guess & Prove combinatorial walks
3. Proofs of identities
4. Sums and Integrals
5. Numerical evaluation via the Taylor series
6. Chebyshev expansions

# I. Equations as a data-structure

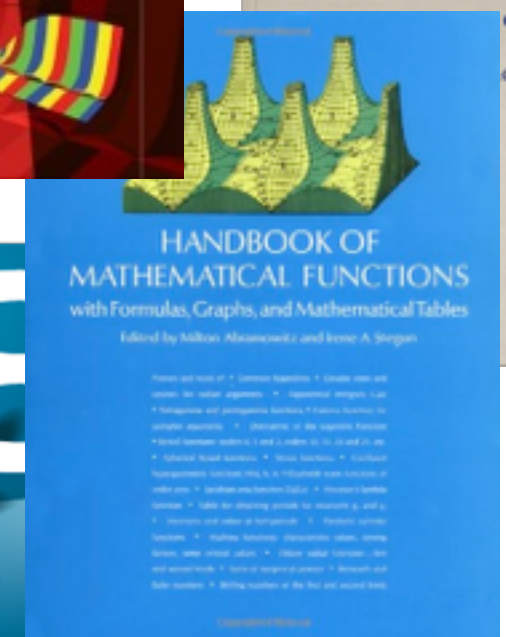
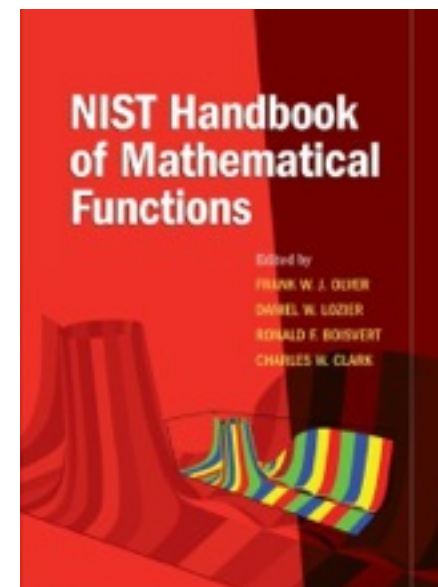
erf := ( $y'' + 2xy' = 0$ , ini. cond.)

basis of the [gfun](#) package

# Dynamic Dictionary of Mathematical Functions

- User need
- Recent algorithmic progress
- Maths on the web

<http://ddmf.msr-inria.inria.fr/>



*Heavy work by F. Chyzak*



# Demonstration

Dynamic Dictionary of Mathematical Functions

ddmf.msr-inria.inria.fr/1.9.1/ddmf

## Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on [Mathematical Functions](#), with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

**This is release 1.9.1 of DDMF**  
**Select a special function from the list**

**What's new?** The main changes in this release 1.9.1, dated May 2013, are:

- Proofs related to Taylor polynomial approximations.

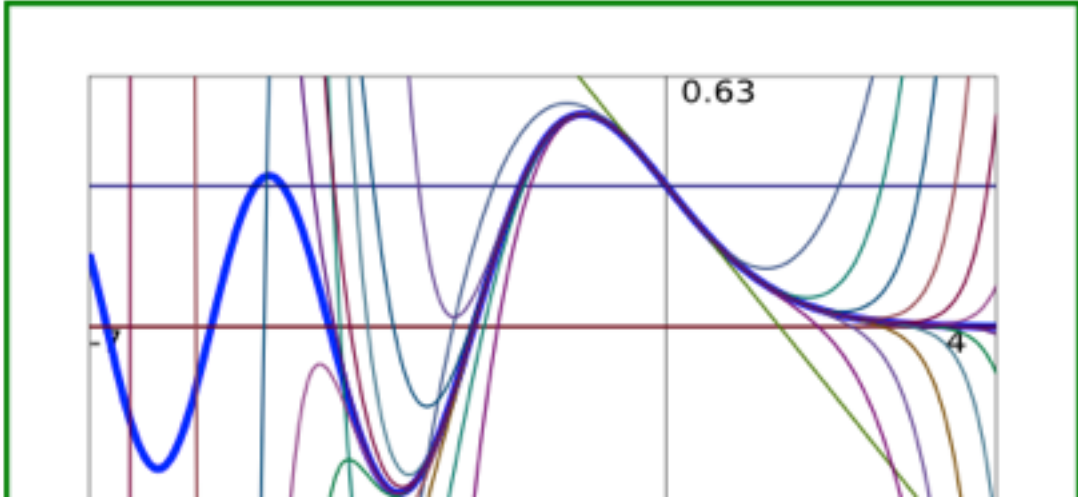
Release [history](#).

**More on the project:**

- [Help](#) on selecting and configuring the mathematical rendering
- [DDMF developers](#) list
- [Motivation](#) of the project
- [Article](#) on the project at ICMS'2010
- [Source code](#) used to generate these pages
- List of [related projects](#)

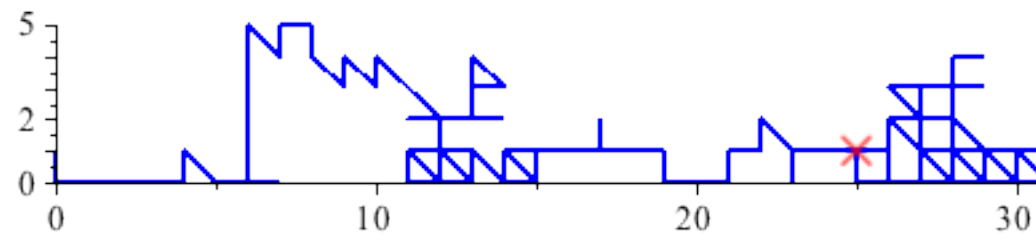
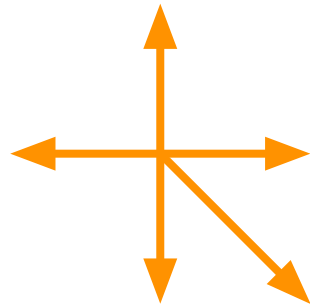
**Mathematical Functions**

- The [Airy function of the first kind](#)  $\text{Ai}(x)$
- The [Airy function of the second kind](#)  $\text{Bi}(x)$
- The [Anger function](#)  $\text{J}_n(x)$
- The [inverse cosine](#)  $\arccos(x)$
- The [inverse hyperbolic cosine](#)  $\text{arccosh}(x)$
- The [inverse cotangent](#)  $\text{arccot}(x)$
- The [inverse hyperbolic cotangent](#)  $\text{arccoth}(x)$
- The [inverse cosecant](#)  $\text{arccsc}(x)$
- The [inverse hyperbolic cosecant](#)  $\text{arccsch}(x)$
- The [inverse secant](#)  $\text{arcsec}(x)$
- The [inverse hyperbolic secant](#)  $\text{arcsech}(x)$
- The [inverse sine](#)  $\arcsin(x)$
- The [inverse hyperbolic sine](#)  $\text{arcsinh}(x)$
- The [inverse tangent](#)  $\arctan(x)$
- The [inverse hyperbolic tangent](#)  $\text{arctanh}(x)$
- The [modified Bessel function of the first kind](#)  $I_\nu(x)$
- The [Bessel function of the first kind](#)  $J_\nu(x)$
- The [modified Bessel function of the second kind](#)  $K_\nu(x)$
- The [Bessel function of the second kind](#)  $Y_\nu(x)$
- The [Chebyshev function of the first kind](#)  $T_n(x)$
- The [Chebyshev function of the second kind](#)  $U_n(x)$
- The [hyperbolic cosine integral](#)  $\text{Chi}(x)$
- The [cosine integral](#)  $\text{Ci}(x)$
- The [cosine](#)  $\cos(x)$
- The [hyperbolic cosine](#)  $\cosh(x)$
- The [Coulomb function](#)  $F_n(l, x)$
- The [Whittaker's parabolic function](#)  $D_a(x)$
- The [parabolic cylinder function](#)  $U(a, x)$
- The [parabolic cylinder function](#)  $V(a, x)$



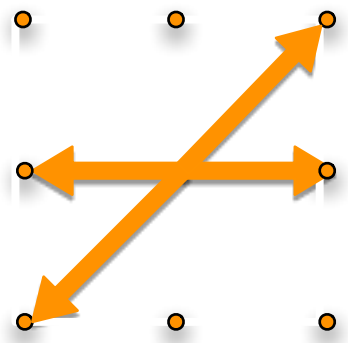
# II. Guess & Prove

## Combinatorial Walks

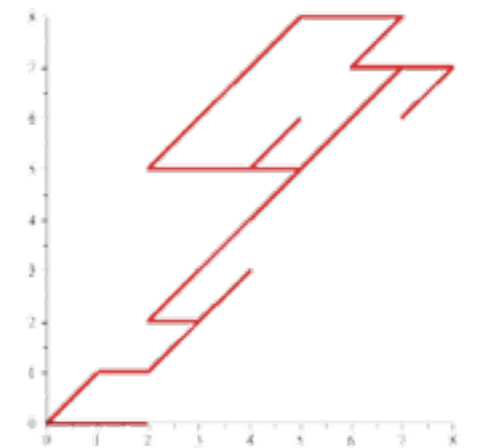


# Gessel's walks in the 1/4-plane

$$G(x, y, t) := \sum_{n \geq 0} \sum_{i, j} f_{i, j; n} x^i y^j t^n$$



- 79 inequivalent step sets (MBM & Mishna);
- long history of special cases;
- Gessel's was left;
- conjectured not soln of LDE.



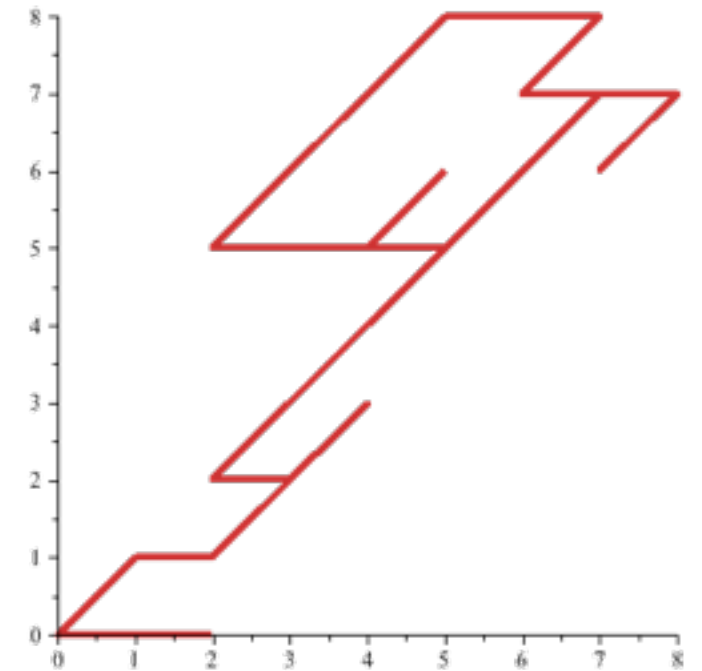
Thm. [Bostan-Kauers 2010]  
G is algebraic!

Computer-driven discovery & proof.



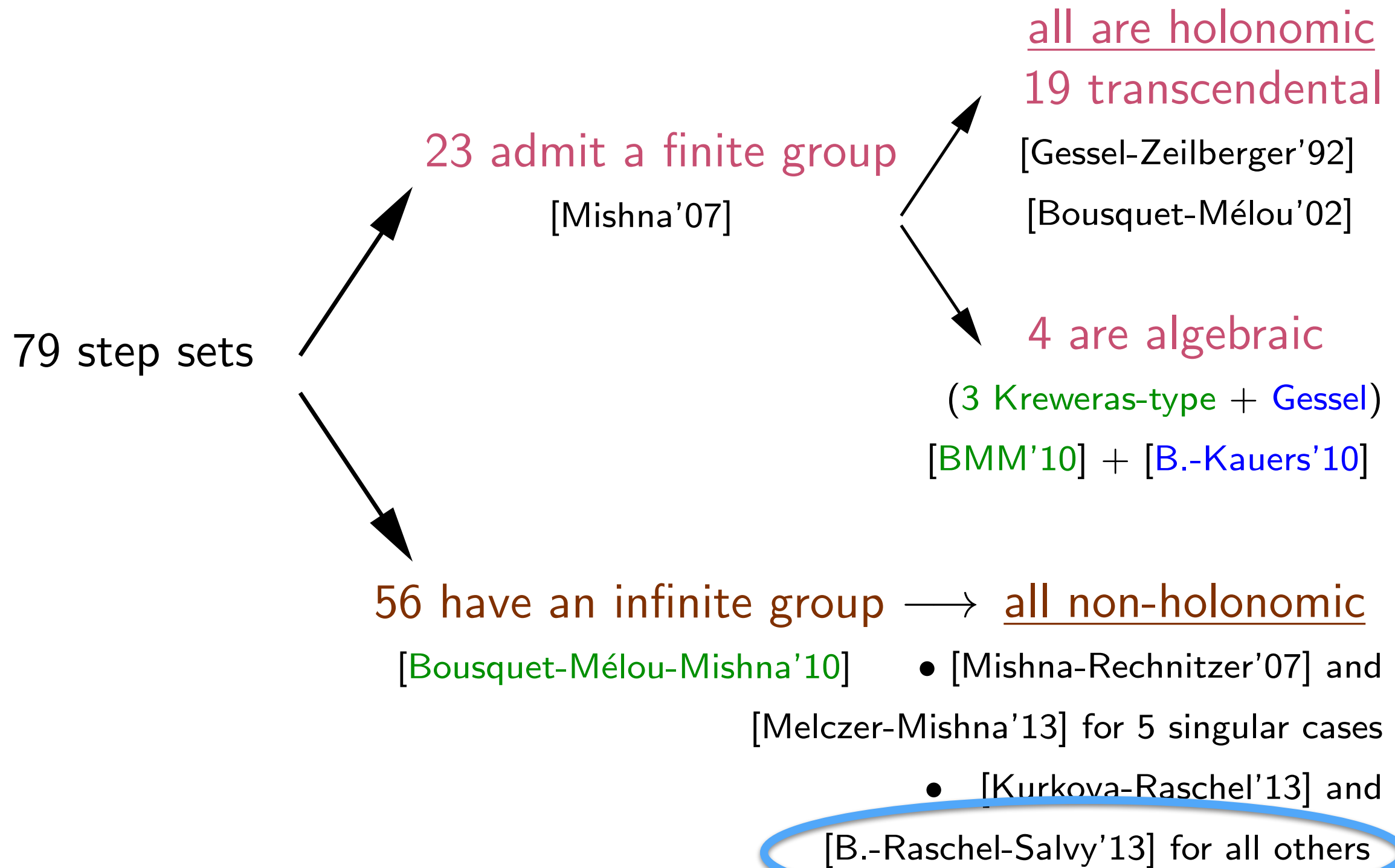
# Computation

$$G(x, y, t) := \sum_{n \geq 0} \sum_{i, j} f_{i, j; n} x^i y^j t^n$$



- Compute  $G$  up to  $t^{1000}$ ;
- conjecture LDE with 1.5 million coeffs!
- check for sanity (bit size, more coeffs, Fuchsian,  $p$ -curvature);
- oh!
- conjecture polynomials ( $\deg \leq (45, 25, 25)$ , 25 digit coeffs);
- **Proof** by (big) resultants.

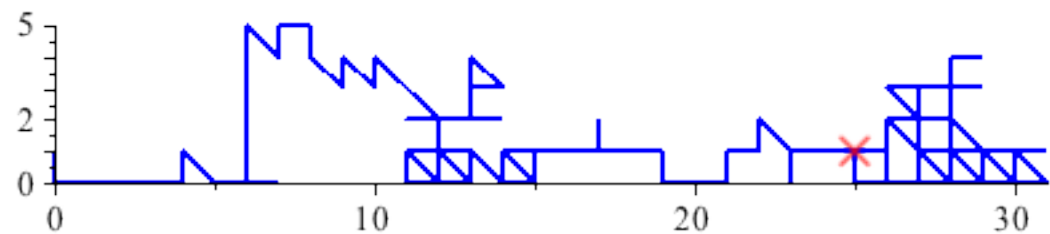
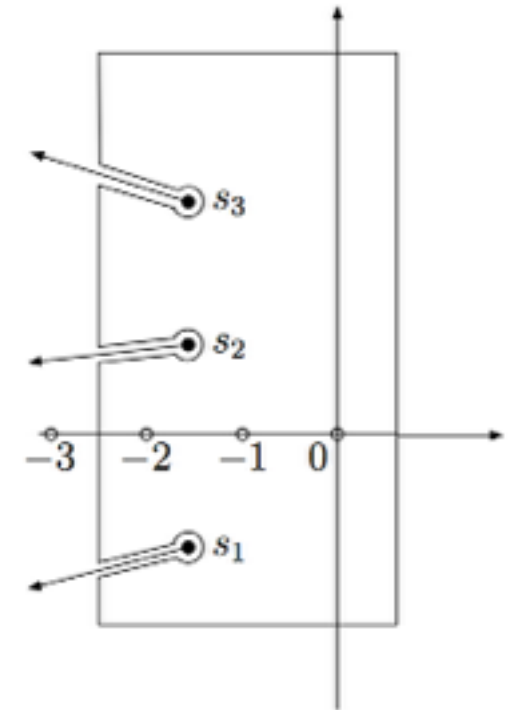
# The 79 cases: finite and infinite groups



*[Slide borrowed from A. Bostan]*

# Proving no LDE exists

- $\log(n)$ ,  $n^a$  ( $a \notin \mathbf{Z}$ ),  $p_n$ ,  $e^{\sqrt{n}}$ ,  $\Gamma(n\sqrt{2})$ , ...  
by their asymptotic behavior,  
or that of their generating function  
(with P. Flajolet & S. Gerhold).

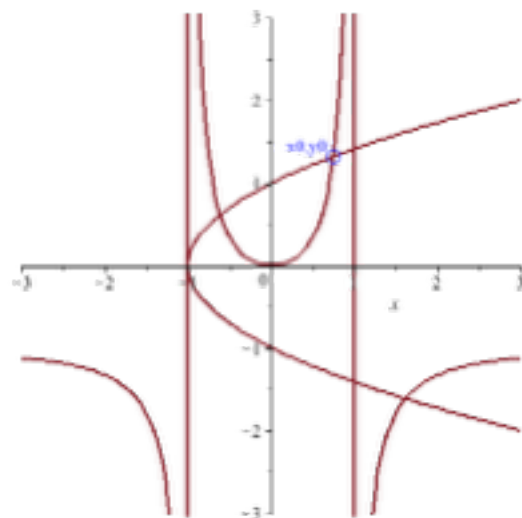
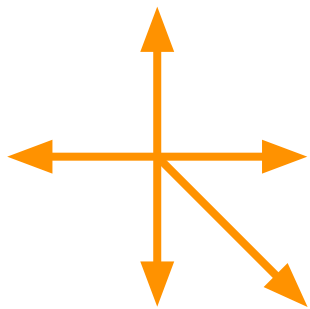


- walks in the 1/4-plane  
(with A. Bostan & K. Raschel)

$$\chi := \sum_{(i,j) \in \mathfrak{G}} x^i y^j$$

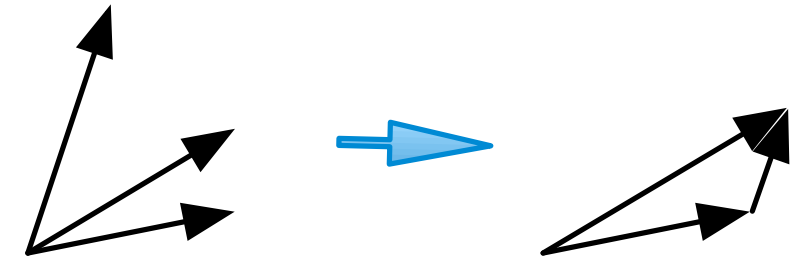
$$c := \frac{\frac{\partial^2 \chi}{\partial x \partial y}}{\sqrt{\frac{\partial^2 \chi}{\partial x^2} \cdot \frac{\partial^2 \chi}{\partial y^2}}} (x_0, y_0)$$

$$\arccos(c)/\pi \notin \mathbb{Q}$$



# III. Proofs of Identities

# Proof technique



> `series(sin(x)^2+cos(x)^2-1,x,4);`

f satisfies a LDE



f, f', f'', ... live in a  
finite-dim. vector space

$O(x^4)$

Why is this a proof?

1. sin and cos satisfy a 2nd order LDE:  $y''+y=0$ ;
2. their squares and their sum satisfy a 3rd order LDE;
3. the constant -1 satisfies  $y'=0$ ;
4. thus  $\sin^2+\cos^2-1$  satisfies a LDE of order at most 4;
5. Cauchy's theorem concludes.

Proofs of non-linear identities by linear algebra!

# Example: Mehler's identity for Hermite polynomials

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{u^n}{n!} = \frac{\exp\left(\frac{4u(xy - u(x^2 + y^2))}{1 - 4u^2}\right)}{\sqrt{1 - 4u^2}}$$

1. Definition of Hermite polynomials:  
recurrence of order **2**;
2. Product by linear algebra:  $H_{n+k}(x)H_{n+k}(y)/(n+k)!$ ,  $k \in \mathbb{N}$   
generated over  $\mathbb{Q}(x, n)$  by

$$\frac{H_n(x)H_n(y)}{n!}, \frac{H_{n+1}(x)H_n(y)}{n!}, \frac{H_n(x)H_{n+1}(y)}{n!}, \frac{H_{n+1}(x)H_{n+1}(y)}{n!}$$

→ recurrence of order **at most 4**;

3. Translate into differential equation.

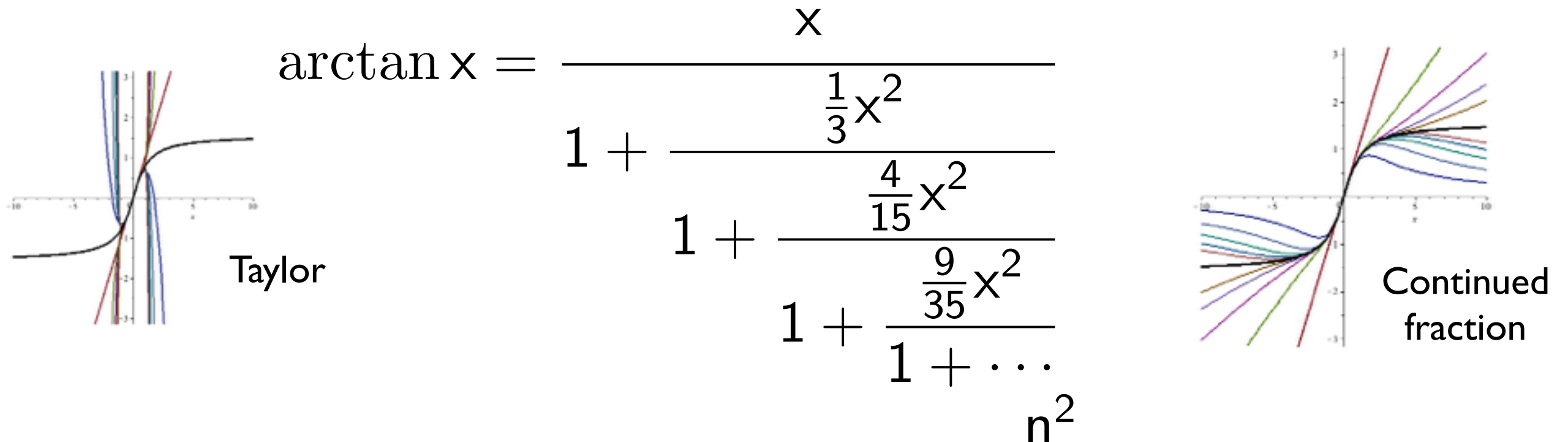




# Guess & prove continued fractions

(with S. Maulat)

- Differential equation produces first terms (easy):



- Guess** a formula (easy):  $a_n = \frac{n^2}{4n^2 - 1}$
- Prove** that the CF with these  $a_n$  satisfies the differential equation.

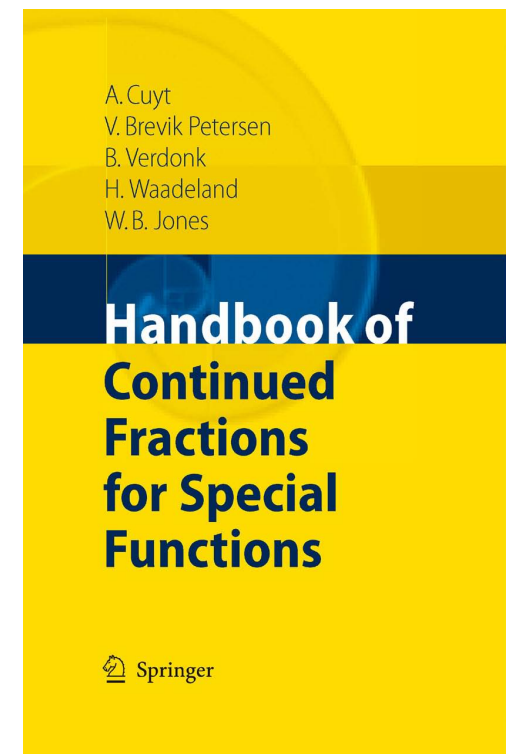
No human intervention needed.

# Automatic Proof of the guessed CF

$$\arctan x \stackrel{?}{=} \frac{x}{1 + \frac{\dots}{1 + \frac{\frac{n^2}{4n^2 - 1} x^2}{1 + \dots}}}$$

- **Aim:** RHS satisfies  $(x^2+1)y'-1=0$ ;
- Convergents  $P_n/Q_n$  where  $P_n$  and  $Q_n$  satisfy a LRE (and  $Q_n(0) \neq 0$ );
- Define  $H_n := (Q_n)^2((x^2+1)(P_n/Q_n)' - 1)$ ;
- $H_n$  is a polynomial in  $P_n, Q_n$  and their derivatives;
- **therefore**, it satisfies a LRE that can be computed;
- from it,  $H_n = O(x^n)$  visible, ie  $\lim P_n/Q_n$  soln;
- **conclude**  $P_n/Q_n \rightarrow \arctan$  (check initial cond.).

More generally: this guess-and-proof approach applies to CF for solutions of (q-)Ricatti equations  
 → all explicit C-fractions in Cuyt et alii.



# IV. Sums and Integrals

# Creative telescoping

$$I(\boldsymbol{x}) = \int f(\boldsymbol{x}, t) dt =? \quad \text{or} \quad S(n) = \sum_k u(n, k) =?$$

**Input:** equations  
(differential for  $f$  or  
recurrence for  $u$ ).

**Output:** equations for the  
sum or the integral.

**Method:** integration (summation) by parts and differentiation  
(difference) under the integral (sum) sign

Example (with Pascal's triangle):

$$u(n, k) = \binom{n}{k} \text{ def. by } \left\{ \binom{n+1}{k} = \frac{n+1}{n+1-k} \binom{n}{k}, \binom{n}{k+1} = \frac{n-k}{k+1} \binom{n}{k} \right\}$$

$$S(n+1) = \sum_k \binom{n+1}{k} = \sum_k \underbrace{\binom{n+1}{k} - \binom{n+1}{k+1}}_{\text{telesc.}} + \underbrace{\binom{n}{k+1} - \binom{n}{k}}_{\text{telesc.}} + 2\binom{n}{k} = 2S(n).$$

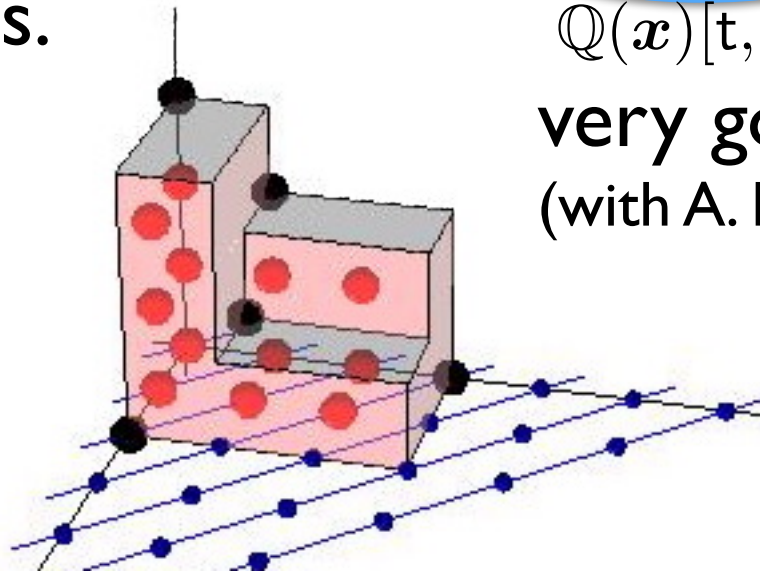
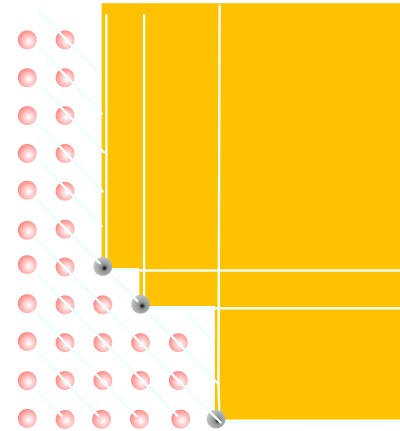
**Algorithms:** reduce the search space.

# Telescoping Ideal

$$T_t(f) := \left( \text{Ann } f + \underbrace{\partial_t \mathbb{Q}(\mathbf{x}, t) \langle \partial_{\mathbf{x}}, \partial_t \rangle}_{\text{int. by parts}} \right) \cap \underbrace{\mathbb{Q}(\mathbf{x}) \langle \partial_{\mathbf{x}} \rangle}_{\text{diff. under } f} .$$

- hypergeometric summation:  
dim=1 + param. Gosper.  
[Zeilberger]
- holonomy: restrict int.  
by parts to  $\mathbb{Q}(\mathbf{x}) \langle \partial_{\mathbf{x}}, \partial_t \rangle$   
and Gröbner bases.  
[Wilf-Zeilberger]

- finite dim, Ore algebras  
& GB (with F. Chyzak)
- infinite dim & GB  
(with F. Chyzak & M. Kauers)
- **rational  $f$**  and restrict to  
 $\mathbb{Q}(\mathbf{x})[t, 1/\text{den } f] \langle \partial_{\mathbf{x}}, \partial_t \rangle$  in  
**very good complexity**  
(with A. Bostan & P. Lairez)



# Multiple binomial sums

(with A. Bostan & P. Lairez)

**Def.** Combination of binomial coefficients, geometric sequences,  $+$ ,  $\times$ , multiplication by scalars, affine changes of indices and  $\sum$ .

**Thm.** The *generating series* is the integral of a rational function that can be constructed automatically.

→ whence an efficient summation algorithm using the previous algo.

**Ex. (Apéry)**  $A(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \mapsto \oint \frac{dt_1 \wedge dt_2 \wedge dt_3}{t_1 t_2 t_3 (1 - t_1 t_2 - t_1 t_2 t_3) - (1 + t_1)(1 + t_2)(1 + t_3)z}$   
 $\mapsto \text{LDE} \mapsto (n+1)^3 A(n) + (\dots) A(n+1) + (n+2)^2 A(n+2) = 0.$

More at Pierre's PhD defense on **Nov. 12.**



# V. Numerical evaluation via the Taylor series

*From large integers to precise numerical values*

# Numerical evaluation of solutions of LDEs

**Principle:**  $f(x) = \underbrace{\sum_{n=0}^N a_n x^n}_{\text{fast evaluation}} + \underbrace{\sum_{n=N+1}^{\infty} a_n x^n}_{\text{good bounds}}$

f solution of a LDE with coeffs in  $\mathbb{Q}(x)$  (our data-structure!)

1. linear recurrence in  $N$  for the first sum (easy);
2. tight bounds on the tail (e.g., work with M. Mezzarobba);
3. no numerical roundoff errors.

The technique used for fast evaluation of constants like

$$\frac{1}{\pi} = \frac{12}{C^{3/2}} \sum_{n=0}^{\infty} \frac{(-1)^n (6n)! (A + nB)}{(3n)! n!^3 C^{3n}} \quad \begin{array}{l} \text{with } A=13591409, \\ B=545140134, \\ C=640320. \end{array}$$

Code available: [NumGfun](#) [Mezzarobba 2010]

# Binary Splitting for linear recurrences (70's and 80's)

- $n!$  by divide-and-conquer:

$$n! = \underbrace{n \times \cdots \times \lceil n/2 \rceil}_{\text{size } O(n \log n)} \times \underbrace{\lfloor n/2 \rfloor \times \cdots \times 1}_{\text{size } O(n \log n)}$$

Cost:  $O(n \log^3 n \log \log n)$  using FFT

- linear recurrences of order 1 reduce to

$$p!(n) := (p(n) \times \cdots \times p(\lceil n/2 \rceil)) \times (p(\lfloor n/2 \rfloor) \times \cdots \times p(1))$$

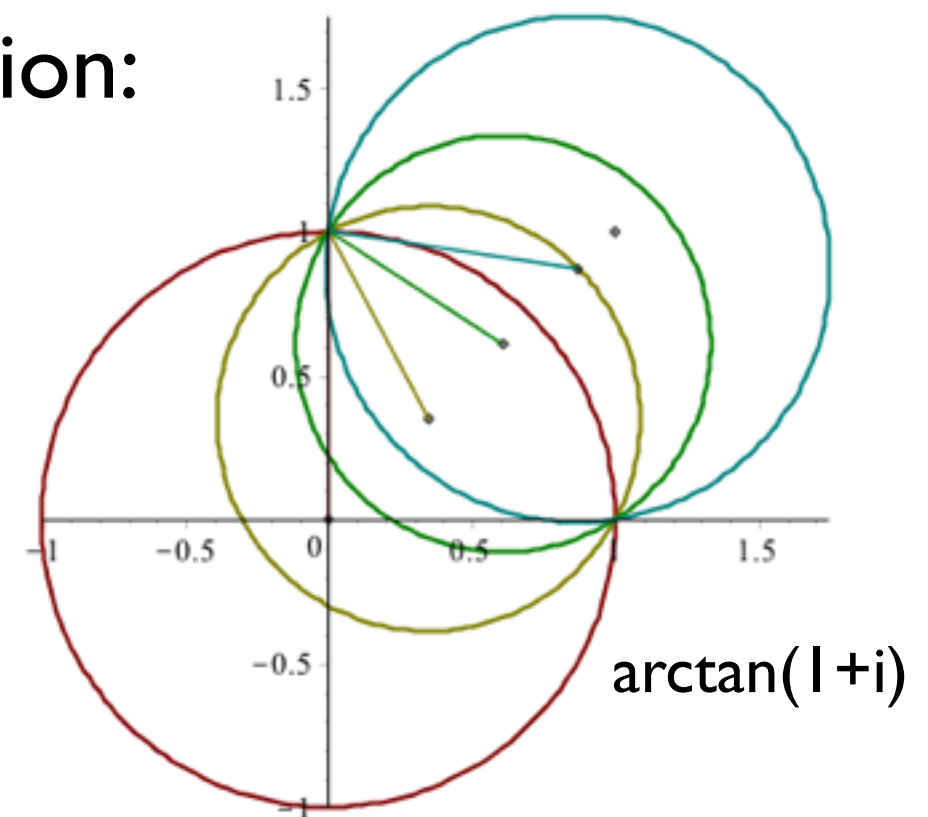
- arbitrary order: same idea, same cost (matrix factorial):

ex:  $e_n := \sum_{k=0}^n \frac{1}{k!}$  satisfies a 2nd order rec, computed via

$$\begin{pmatrix} e_n \\ e_{n-1} \end{pmatrix} = \frac{1}{n} \underbrace{\begin{pmatrix} n+1 & -1 \\ n & 0 \end{pmatrix}}_{A(n)} \begin{pmatrix} e_{n-1} \\ e_{n-2} \end{pmatrix} = \frac{1}{n!} A!(n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

# Analytic continuation

Compute  $f(x), f'(x), \dots, f^{(d-1)}(x)$  as new initial conditions and handle error propagation:

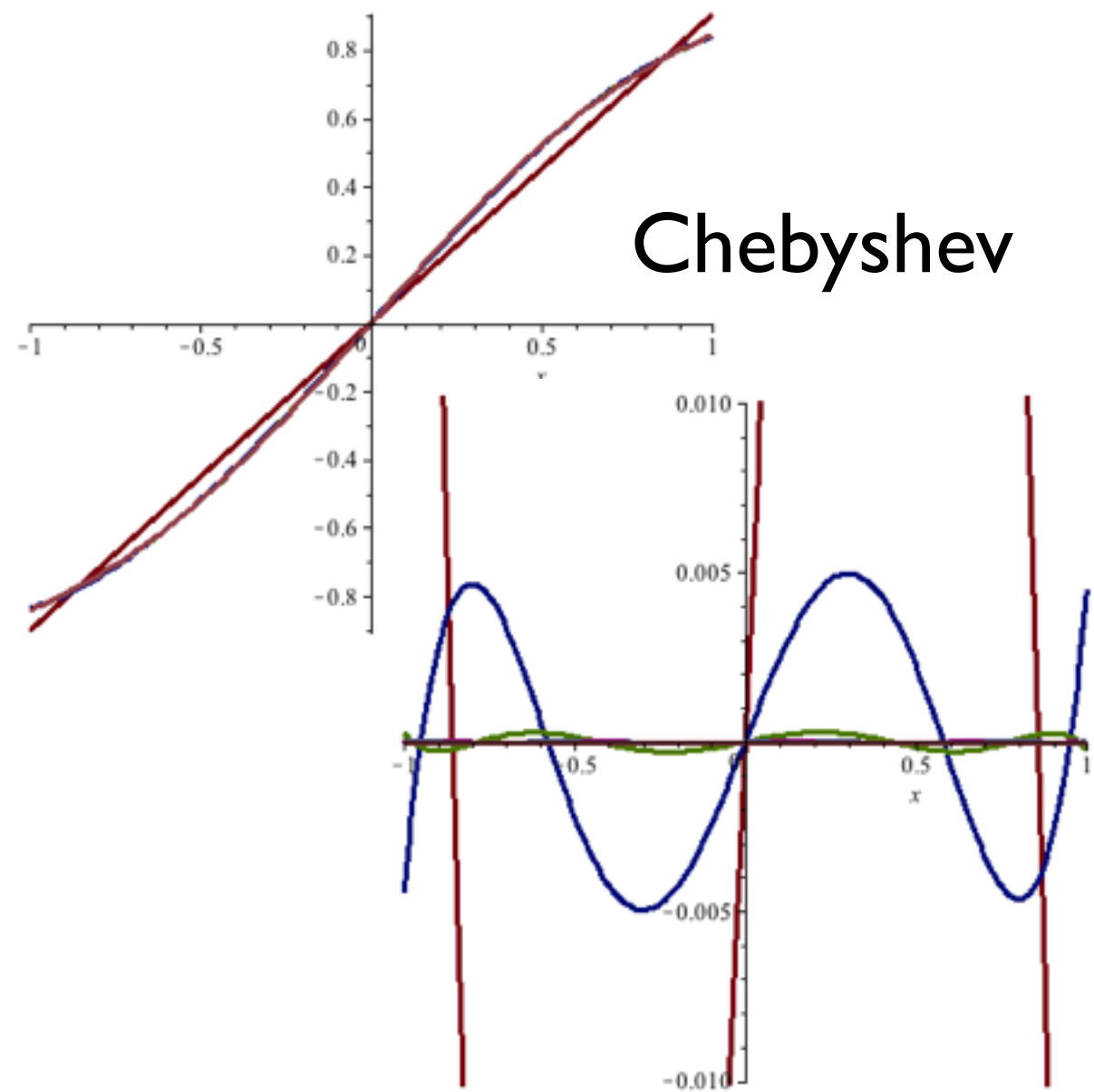
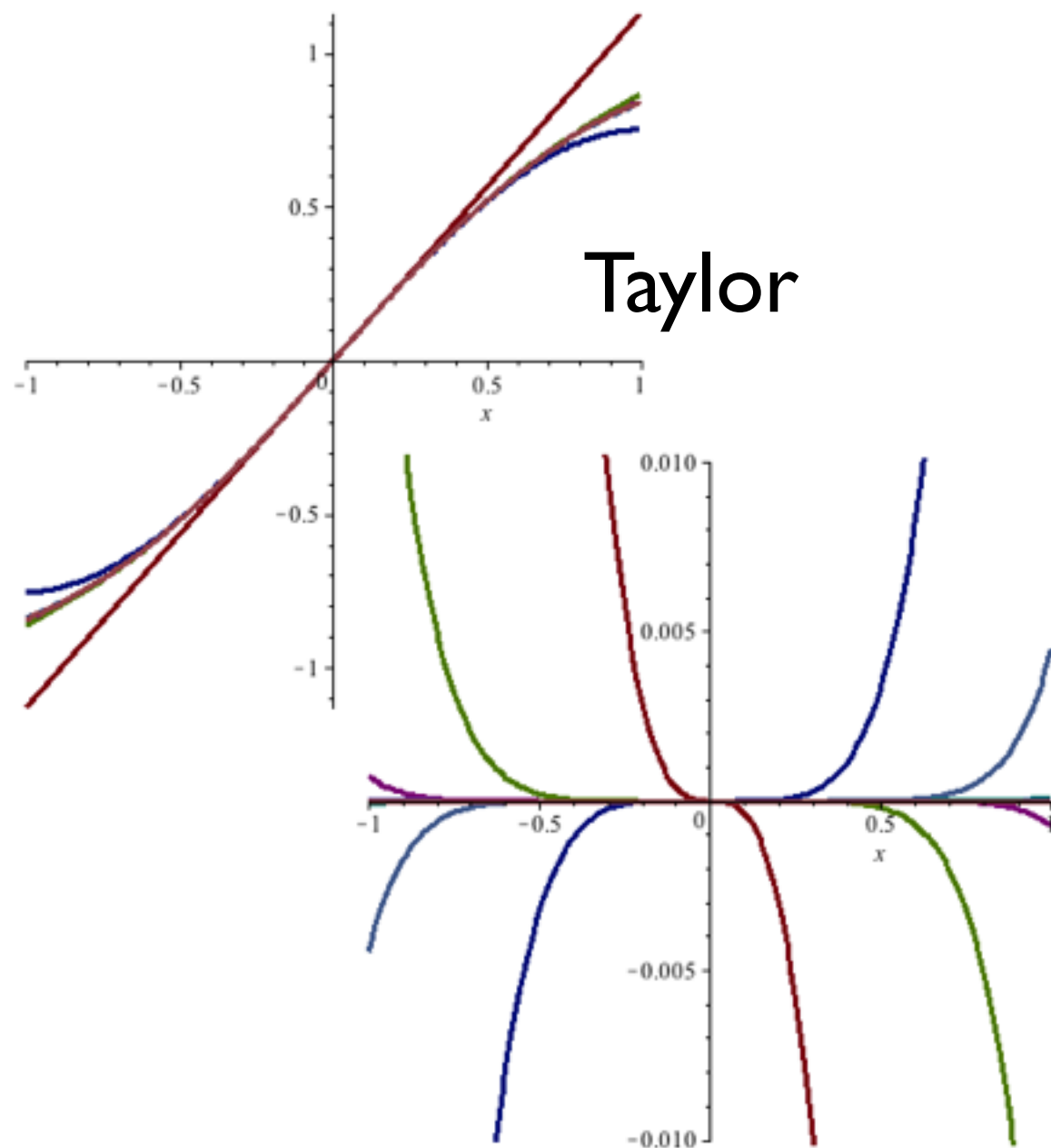


Ex:  $\text{erf}(\pi)$  with 15 digits:

$$0 \xrightarrow[200 \text{ terms}]{} 3.1416 \xrightarrow[18 \text{ terms}]{} 3.1415927 \xrightarrow[6 \text{ terms}]{} 3.14159265358979$$

Again: computation on integers. No roundoff errors.

# VI Chebyshev expansions



# From equations to operators

$d/dx \leftrightarrow D$   
mult by  $x \leftrightarrow x$   
composition  $\leftrightarrow$  product  
 $Dx = xD + I$

$(n \mapsto n+1) \leftrightarrow S$   
mult by  $n \leftrightarrow n$   
composition  $\leftrightarrow$  product  
 $S_n = (n+1)S$

**Taylor morphism:**  $D \mapsto (n+1)S$ ;  $x \mapsto S^{-1}$   
produces linear recurrence from LDE

erf:  $D^2 + 2xD \mapsto (n+1)S(n+1)S + 2S^{-1}(n+1)S = (n+1)(n+2)S^2 + 2n$

Ore (1933): general framework for these non-commutative polynomials.

**Main property:**  $\deg AB = \deg A + \deg B$ .

**Consequence 1:** (non-commutative) **Euclidean division**

**Consequence 2:** (non-commutative) **Euclidean algorithm.**



# Ore fractions ( $Q^{-1}P$ with P&Q operators)

Thm. (Ore 1933) Sums and products reduce to that form.

Application: extend Taylor morphism to Chebyshev expansions

Taylor

$$x^{n+1} = x \cdot x^n, (x^n)' = nx^{n-1}$$

$$\leftrightarrow X := S^{-1}, D := (n+1)S$$

Chebyshev

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x),$$

$$2(1-x^2)T_n'(x) = -nT_{n+1}(x) + nT_{n-1}(x)$$

$$\leftrightarrow X := (S+S^{-1})/2,$$

$$D := (1-X^2)^{-1}n(S-S^{-1})/2 = 2(S^{-1}-S)^{-1}n.$$

erf:  $D^2 + 2xD \mapsto (2(S^{-1} - S)^{-1}n)^2 + 2 \frac{S + S^{-1}}{2} 2(S^{-1} - S)^{-1}n$

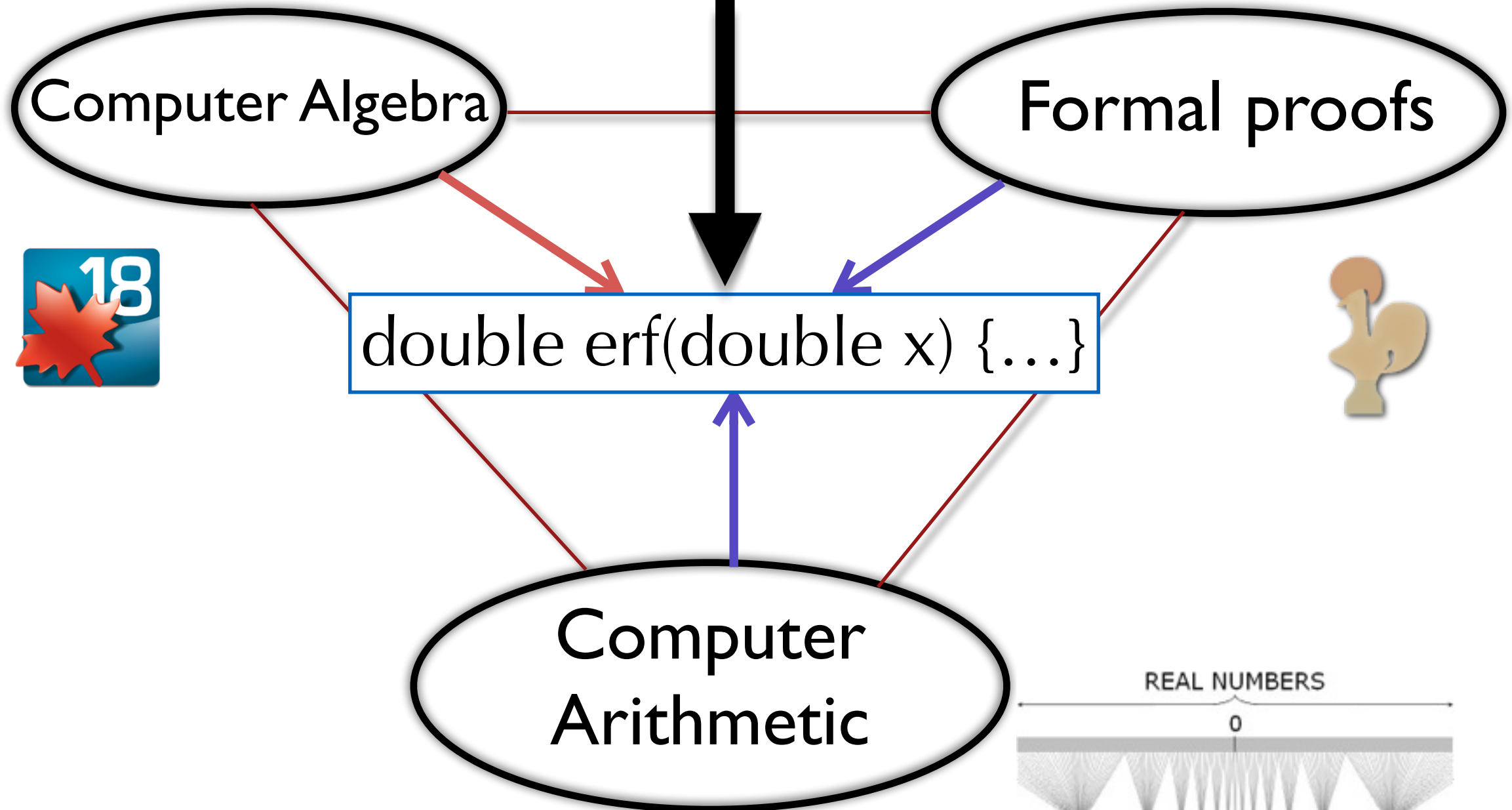
$$= \text{pol}(n, S)^{-1} (2(n+1)(n+4)S^4 - 4(n+2)^3S^2 + 2n(n+3))$$

**Prop.** [with A. Benoit] If  $y$  is a solution of  $L(x, d/dx)$ , then its Chebyshev coefficients annihilate the **numerator** of  $L(X, D)$ .

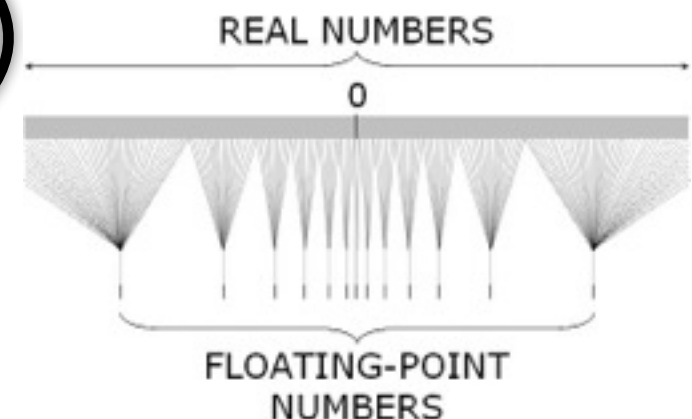
Efficient numerical use: [arXiv:1407.2802](https://arxiv.org/abs/1407.2802)

# Next steps: FastRelax (starting this Fall)

$$y'' + 2xy' = 0 + \text{ini. cond.}$$



5 teams, 4 years



# Conclusion

- Linear differential equations and recurrences are a great data-structure;
- Numerous algorithms have been developed in computer algebra;
- Efficient code is available;
- More is true (creative telescoping, diagonals,...);
- More to come for Mireille and her followers.

Bravo Mireille !